Paradoxes in complex number for beginners

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Here is something which may perplex you if you meet the concept of complex numbers for awhile.

Problem 1

Observe that:

$$(-1)^{\frac{3}{2}} = [(-1)^3]^{\frac{1}{2}} = [-1]^{\frac{1}{2}} = \mathrm{i} \qquad \ldots.(1)$$

$$(-1)^{\frac{3}{2}} = \left[(-1)^{\frac{1}{2}}\right]^3 = [\mathrm{i}]^3 = -\mathrm{i} \qquad \ldots.(2)$$

Now, adding (1) and (2) and then divide by 2, we get:

$$(-1)^{\frac{3}{2}} = 0$$

Is it correct?

The following is a calculation of an expression in two ways, and they show the same answer. Do you think it is correct? Is it good for you to use your ***calculator*** to "confirm" it?

Problem 2

Observe that:

$$(-1)^{\frac{2}{3}} = \left[(-1)^{\frac{1}{3}}\right]^2 = [-1]^2 = 1 \qquad \ldots.(3)$$

$$(-1)^{\frac{2}{3}} = [(-1)^2]^{\frac{1}{3}} = [1]^{\frac{1}{3}} = 1 \qquad \ldots.(4)$$

$$\therefore \quad (-1)^{\frac{2}{3}} = 1$$

Is it right?

Is it safe for us to say:

Problem 3

$$(-1)^{\frac{4}{3}} = (-1)^{\frac{2}{3}} = 1 \qquad \ldots.(5)$$

How can you evaluate:

Problem 4

$$(-1)^{\frac{1}{4}}$$

All the above problems can be easily solved with the help of **de'Moivres Theorem** and/or **Euler formula**. But for beginners, let us use a powerful tool in algebra to tackle them that is, solving equations.

Problem 1

Let $z = (-1)^{\frac{3}{2}}$

Then $z^2 = (-1)^3 = -1$

We get an equation: $z^2 + 1 = 0$ .....(6)

$(z + i)(z - i) = 0$

We have: $z = (-1)^{\frac{3}{2}} = \pm i$

We therefore say that $(-1)^{\frac{3}{2}}$ is no longer a single-valued expression. It has two possible answers: +i or –i .

Let us continue with our adventure to **Problem 2**. The solution is quite unexpected.

Problem 2

Let $z = (-1)^{\frac{2}{3}}$

$z^3 = (-1)^2 = 1$

$z^3 - 1 = 0$ ..... (7)

$(z - 1)(z^2 + z + 1) = 0$

$z = 1$ or $z^2 + z + 1 = 0$

$z = 1, \frac{-1 \pm \sqrt{-3}}{2}$ (Quadratic equation formula)

$\therefore z = (-1)^{\frac{2}{3}} = 1, \frac{-1-\sqrt{3}i}{2}, \frac{-1+\sqrt{3}i}{2}$

Hence $(-1)^{\frac{2}{3}}$ has 3 possible answers.

Problem 3 seems to be easy, but checking still needs some knowledge of algebra.

Problem 3

Let $$z = (-1)^{\frac{4}{3}}$$

$$z^3 = (-1)^4 = 1$$

$$z^3 - 1 = 0 \qquad \ldots\ldots(8)$$

But equation (8) is the same as equation (7), so they share the same roots.

$$(-1)^{\frac{4}{3}} = (-1)^{\frac{2}{3}} = 1, \quad \frac{-1 \pm \sqrt{3}i}{2}$$

Starting from the result in **Problem 3**, you can also check that:

$$(-1)^{\frac{4}{3}} = \left[(-1)^{\frac{2}{3}}\right]^2 = 1^2, \left(\frac{-1 \pm \sqrt{3}i}{2}\right)^2$$

$$= 1, \ \frac{-1 \mp \sqrt{3}i}{2}$$

There is not much trouble in setting the equation in **Problem 4**, but solving the equation is another matter.

Problem 4

Let $$z = (-1)^{\frac{1}{4}}$$

$$z^4 = -1$$

$$z^4 + 1 = 0 \qquad \ldots\ldots(9)$$

To solve equation (9), we need to insert a term for completing square.

$$(z^4 + 2z^2 + 1) - 2z^2 = 0$$

$$(z^2 + 1)^2 - \left(\sqrt{2}z\right)^2 = 0$$

$$\left(z^2 + \sqrt{2}z + 1\right)\left(z^2 - \sqrt{2}z + 1\right) = 0$$

$$z^2 + \sqrt{2}z + 1 = 0 \quad \text{or} \quad z^2 - \sqrt{2}z + 1 = 0$$

"Unusual" factorization with irrational $\sqrt{2}$!

Using quadratic equation formula, we get 4 answers:

$$z = (-1)^{\frac{1}{4}} = \frac{-\sqrt{2} \pm \sqrt{2}i}{2}, \quad \frac{\sqrt{2} \pm \sqrt{2}i}{2} = \frac{\pm\sqrt{2} \pm \sqrt{2}i}{2}$$

More to ponder

$$z = (1)^{\frac{1}{2}} \Rightarrow z^2 = 1 \Rightarrow z^2 - 1 = 0 \Rightarrow z = \pm 1 \Rightarrow (1)^{\frac{1}{2}} = \pm 1$$